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From $\mathrm{AdS}_3/\mathrm{CFT}_2$ to black holes/topological strings

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Abstract: The elliptic genus Z_{BH} of a large class of 4D black holes can be expressed as an M-theory partition function on an $AdS_3 \times S^2 \times CY_3$ attractor. We approximate this partition function by summing over multiparticle chiral primary states of membranes which wrap curves in the CY_3 and tile Landau levels on the horizon S^2. Significantly, membranes and antimembranes can preserve the same supercharges if they occupy antipodal points on the horizon. It is shown the membrane contribution to Z_{BH} gives precisely the topological string partition function Z_{top} while the antimembranes give $\bar{Z}_{\mathrm{top}}$, implying $Z_{BH} = |Z_{\mathrm{top}}|^2$ in this approximation.

Keywords: Black Holes in String Theory, Topological Strings.

Contents

1. Introduction

A four-dimensional black hole in an M-theory compactification on a Calabi-Yau threefold X times an S^1 can be constructed by wrapping an M5-brane with fluxes and S^1 momentum on a 4-cycle P in X. This black hole is dual to the R sector of the $(0, 4)$ CFT which lives on the dimensionally reduced M5 worldvolume [1]. The NS sector of this same $(0, 4)$ CFT is dual to supergravity on $AdS_3 \times S^2 \times X$ (as well as the 5D black ring [2]).

The elliptic genus $Z_{BH} = Z_{CFT}$ of the 4D black hole is[1]

$$Z_{BH} = \mathrm{Tr}(-)^F q^{L_0 - \frac{c_L}{24}} e^{2\pi i q_A y^A}, \tag{1.1}$$

where the trace is over chiral primary states on $AdS_3 \times S^2 \times X$, q_A is a membrane charge and y^A the conjugate potential. We work in the dilute gas expansion in which (1.1) is dominated by multi-particle chiral primaries states of membranes wrapping holomorphic curves in X.

A crucial point in the following is that both membrane and anti-membrane states contribute to (1.1). This is because a membrane wrapping a holomorphic curve $Q = q_A \alpha^A$ in X and sitting at say the north pole of the S^2 preserves the the *same* set of supersymmetries as the oppositely-charged *anti*-M2-brane wrapping Q and sitting at the

[1] This is related by spectral flow to the Ramond sector trace. If the center of mass multiplet is included, there should be an extra insertion of F^2/τ_2, but we will suppress this herein.

south pole [3]. This may sound strange as we are used to the idea in flat space that branes and antibranes preserve opposite supersymmetries because they have opposite charges. However in $AdS_2 \times S^2$ the S^2 angular momentum plays the role of the central charge in stabilizing BPS states. Static wrapped branes in this background carry this angular momentum much like a static electron in the field of a monopole. Hence branes and antibranes at anitpodal points can carry the same angular momentum and preserve the same supersymmetries.

In this paper we work out in detail the degeneracies of chiral primary wrapped membranes of all stripes and their contribution to Z_{BH}. We find the product of two complex conjugate factors, one from branes and another from antibranes. Including an additional factor from massless supergravity modes, a modular transformation factor and using the Gopakumar-Vafa relation [4] between BPS degeneracies and Gromov-Witten invariants, we recover precisely the OSV relation [5][2]

$$Z_{BH} = |Z_{\rm top}|^2. \tag{1.2}$$

In [5], this perturbative factorization was discovered by direct brute force computation. In the present work we have found a simple physical explanation: $Z_{\rm top}$ is the membrane contribution, while $\bar{Z}_{\rm top}$ is the anti-membrane contribution.[3]We further hope that the the framework can be used to systematically compute non-perturbative corrections to (1.2), such as perhaps arising from chiral primary wrapped M5 branes.

This paper is organized as follows. Section 2 recaps the M-theory attractor $AdS_3 \times S^2 \times X$. The classical and quantum BPS states of wrapped M2-branes are described in section 3. In section 4 we compute the elliptic genus from the bulk theory, including the contribution from wrapped M2-branes, massless bulk supergravity fields and boundary singletons. The result is compared to the black hole partition function and topological strings in section 5. Appendix A reproduces a useful resummation formula involving the Gopakumar-Vafa invariants.

2. Preliminaries

We consider M-theory on an $AdS_3 \times S^2 \times X$ attractor geometry, where X is a Calabi-Yau threefold, with 4-form flux

$$G_4 = \omega_{S^2} \wedge p^A \omega_A \tag{2.1}$$

Here ω_A is a basis of harmonic 2-forms dual to 2-cycles α^A in X with intersection numbers $\int_X \omega_A \wedge \omega_B \wedge \omega_C = 6D_{ABC}$. The metric is:[4]

$$\int_{\alpha^A} J = (2\pi)^2 \frac{p^A}{\ell} \tag{2.2}$$

$$ds_{11}^2 = \ell^2(-\cosh^2\chi d\tau^2 + d\chi^2 + \sinh^2\chi d\psi^2) + \frac{\ell^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) + ds_X^2, \tag{2.3}$$

[2]The agreement is up to factors which depend only on q and not any of the Calabi-Yau data.

[3]Related discussions of this factorization appear in [6 – 8]–[9] .

[4]We adopt 11D Planck units in which, as in [10], the action is $(2\pi)^{-8} \int d^{11}x\sqrt{-g}R + \cdots$.

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where ℓ is the radius of AdS_3 and J is the Kähler form on X . This is the near horizon attractor geometry of an M5-brane wrapped on the 4-cycles $p^A\Sigma_A$ (where Σ_A are a basis of 4-cycles dual to ω_A) in X and forming an extended string in the noncompact five dimensions. One expects M-theory on $AdS_3\times S^2\times X$ to be dual to the $(0,4)$ CFT on the M5-brane world volume dimensionally reduced on P. This CFT has leading order left central charge $c_L = \frac{3\ell}{2G_3} = 6D_{ABC}p^Ap^Bp^C$, where the 3D Newton constant is $G_3 = \frac{16\pi^7}{Vol_X Vol_{S^2}}$ [1].

3. BPS wrapped branes

3.1 Classical

The geometry (2.3) has a classical supersymmetric M2-brane wrapping a holomorphic genus g curve in the class $C = q_A\alpha^A$ and sitting at the center of AdS_3, $\chi = 0$. The kappa-symmetry analysis is very similar to the analysis of D2-branes in [3] and will not be repeated here. It can sit at any point on the S^2, but the unbroken supersymmetries vary as the point moves on the S^2. We will take it to sit at the north pole with $\theta = 0$. There is also an oppositely charged configuration consisting of an anti-membrane at the south pole ($\theta = \pi$) which preserves the same supersymmetries.

Both of these configurations have non zero angular momentum corresponding to ϕ rotations of the S^2 and given by

$$J^3 = \frac{1}{2}q_Ap^A. \tag{3.1}$$

This angular momentum is carried by the fields much as for a monopole-electron pair in 4D. Although the M2 and anti-M2 have opposite charges, they still carry the same sign J^3 because they sit at opposite poles. Since they are static and saturate the BPS bound $L_0 = J^3$, it follows that AdS_3 mass and angular momentum are classically

$$L_0 = \overline{L}_0 = \frac{1}{2}q_Ap^A. \tag{3.2}$$

This agrees with a direct calculation of the mass $M = \ell(L_0 + \bar{L}_0)$ as the membrane tension $\frac{1}{(2\pi)^2}$ times the membrane area $\frac{(2\pi)^2}{\ell}q_Ap^A$.

3.2 Quantum

Quantum mechanically, the M2 fluctuates over the moduli space $\mathcal{M}_C$ of the genus g curve C in X, and has a degeneracy from worldvolume fermion zero modes. The supersymmetric quantum ground states correspond to cohomology classes on $\mathcal{M}_C$ and BPS hypermultiplets in 5D. This problem was studied in the context of compactification to 5D Minkowski space in [4], where the hypermuliplets have SO(4) $\sim$ SU(2)$_L$ $\times$ SU(2)$_R$ spin content

$$\sum N_{Q,j_L,j_R}\left(\left[\left(0,\frac{1}{2}\right)\oplus 2(0,0)\right]\otimes(j_L,j_R)\oplus\left[\left(\frac{1}{2},0\right)\right]\oplus 2(0,0)\right]\otimes(j_R,j_L)\right) \tag{3.3}$$

for some integers N_{Q,j_L,j_R} which depend on X and Q, the homology class of C. The range of j_L is determined by the genus of C, and j_R is related to the weight of Lefschetz action on the moduli space of C [4].

We wish to find the supersymmetric ground states - i.e. chiral primaries- of these hypermultiplets on $AdS_3 \times S^2$. In this case the unbroken global superalgebra is $\mathrm{SU}(1,1|2)$ and the relevant central charge is the angular momentum J^3 rather than the (graviphoton component of the) charge q_A (this problem was considered [11, 12]). Due to its coupling to the 4-form flux (2.1), the M2-brane feels a magnetic field on the S^2 of $B = q_A p^A$ units of flux. This leads to Landau levels on the S^2 which fall into representations of the $\mathrm{SU}(2) \in \mathrm{SU}(1,1|2)$ rotation. The highest weight states arising from a hypermulitplet in the representaion (j_L, j_R) have total spin

$$J^3 = \frac{1}{2} q_A p^A + m_R + m_L + \frac{1}{2} + l, \tag{3.4}$$

where $l \geq 0, \;\; -j_{L,R} \leq m_{L,R} \leq j_{L,R}$, and l is the orbital angular momentum on the S^2. $m_R + m_L$ appears in this expression because the $\mathrm{U}(1) \in \mathrm{SU}(2)$ rotations of S^2 correspond to a diagonal $\mathrm{U}(1)$ rotation in the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ of R^4. The shift of $\frac{1}{2}$ appears because of the tensor product with a spin half hypermultiplet appearing in the definition (3.3).

The BPS chiral primary bound implies that these states have $\bar{L}_0 = J^3$. They are multiplets under the $\mathrm{SL}(2,R)_L$ conformal algebra which commutes with $\mathrm{SU}(1,1|2)$ and acts on the AdS_3 component of the wavefunction. There is one lowest weight state with $L_0 = \bar{L}_0 + m_L - m_R + \frac{1}{2} = \frac{1}{2} q_A p^A + 2m_L + l + 1$ for $-j_L \leq m_L \leq j_L, \;\; -j_R \leq m_R \leq j_R$. $m_L - m_R$ appears in this expression because the $\mathrm{U}(1)$) spatial rotations of AdS_3 correspond to an anti-diagonal $\mathrm{U}(1)$ rotation in the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ of R^4. Each of these has a further tower of chiral primary descendants obtained by acting with L_{-1}.

In summary for every charge q_A (j_L, j_R) hypermultiplet there is one chiral primary with

$$L_0 = \frac{1}{2} q_A p^A + 2m_L + l + 1 + J_\phi \tag{3.5}$$

for every integrally-spaced value of

$$-j_L \leq m_L \leq j_L, \quad -j_R \leq m_R \leq j_R, \quad l \geq 0, \quad J_\phi \geq 0. \tag{3.6}$$

In addition there are antimembrane chiral primaries. M-theory with no branes is invariant under parity P, which we take to interchange the north and south pole of the S^2 . When branes are added it is invariant under CP where C reverses the brane charges. L_0 and J^3 are CP invariant. Hence the action of CP on a chiral primary gives another chiral primary. In the case at hand it turns each of the above M2-brane states into an antipodally located anti-M2-brane state. However these states will contribute differently to the elliptic genus because they have different charges.

4. The elliptic genus on $AdS_3 \times S^2 \times X$

In this section we compute the supergravity elliptic genus from M theory. In the NS sector this is given by[5]

$$Z_{\mathrm{sugra}}(\tau, y^A) = \mathrm{Tr}_{\overline{L}_0 = J^3_R} (-)^F e^{2\pi i \tau L_0 + 2\pi i q_A y^A} \tag{4.1}$$

[5] Note that we are not including the factor of $q^{-\frac{c_L}{24}}$ in Z_{sugra}, and the L_0 entering here arises only the wrapped branes. This factor corresponds to the ground state energy of AdS_3.

We work in the dilute gas approximation in which the density of chiral primaries is low. This is the case for large Im τ and/or large Im y^A. There will be two kinds of contributions, one from wrapped membranes and one from supergravity modes, which are computed in the next two subsections.

4.1 Wrapped membranes

In this subsection we find the chiral primaries corresponding to membranes wrapped on holomorhpic curves in X.

Genus zero. For simplicity let's first consider the case of an isolated rational genus zero curve with degeneracy N_{q_A}, so that there is no internal (j_L, j_R) contribution. Summing over multiparticle states of this variety with weights and multiplicities given in (3.5), (3.6) gives

$$\begin{aligned} Z^0_{\text{sugra}} = & \prod_{q_A, l\geq 0, J_\psi \geq 0} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + l + J_\psi + 1)} e^{2\pi i q_A y^A})^{N_{q_A}} \\ & \times \prod_{q_A, l\geq 0, J_\psi \geq 0} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + l + J_\psi + 1)} e^{-2\pi i q_A y^A})^{N_{q_A}} \end{aligned} \tag{4.2}$$

The first factor comes from M2-branes while the second comes from anti-M2-branes. We can reorganize the product by defining $n = l + J_\psi + 1$

$$\begin{aligned} Z^0_{\text{sugra}}(\tau, y^A) = & \prod_{q_A, n>0} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + n)} e^{2\pi i q_A y^A})^{nN_{q_A}} \\ & \times \prod_{q_A, n>0} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + n)} e^{-2\pi i q_A y^A})^{nN_{q_A}} \end{aligned} \tag{4.3}$$

Higher genus. For general (j_L, j_R), instead of (4.2) we have

$$\begin{aligned} Z^{j_L, j_R}_{\text{sugra}} = & \prod_{q_A, l, J_\psi, m_L, m_R} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + l + J_\psi + 1 + 2m_L)} e^{2\pi i q_A y^A})^{(-)^{2j_R + 2j_L} N_{q_A, j_L, j_R}} \\ & \times \prod_{q_A, l, J_\psi, m_L, m_R} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + l + J_\psi + 1 + 2m_L)} e^{-2\pi i q_A y^A})^{(-)^{2j_R + 2j_L} N_{q_A, j_L, j_R}} \\ = & \prod_{q_A, n, m_L,} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + n + 2m_L)} e^{2\pi i q_A y^A})^{(-)^{2j_R + 2j_L} n(2j_R+1) N_{q_A, j_L, j_R}} \\ & \times \prod_{q_A, n, m_L} (1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + n + 2m_L)} e^{-2\pi i q_A y^A})^{(-)^{2j_R + 2j_L} n(2j_R+1) N_{q_A, j_L, j_R}}, \end{aligned} \tag{4.4}$$

where J_ψ and l are non-negative integers, n is a positive integer and $-j_{L,R} \leq m_{L,R} \leq j_{L,R}$. We will see below that these terms give all the loop contributions of the squared topological string partition function.

4.2 Supergravity modes

The massless spectrum of M-theory compactified on X consists of $n_H = 2(h^{2,1}(X) + 1)$ hypermultiplets, $n_V = h^{1,1}(M) - 1$ vector multiplets, and a graviton multiplet. Their

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spectrum on $AdS_3 \times S^2$ organizes into short representations of $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SU}(1,1|2)$. The corresponding chiral primaries can be labelled by their $(L_0, \bar{L}_0 = J^3_R)$ quantum numbers, with the spectrum:

$$n_H \bigoplus_{l\geq 0} \left(l+1, l+\frac{1}{2}\right) + n_V \bigoplus_{l\geq 0} [(l+1,l+1)+(l+1,l)]$$
$$+ \bigoplus_{l\geq 0} [(l+1,l+2)+(l+1,l+1)+(l+1,l)+(l+2,l)] \tag{4.5}$$

The spectrum is obtained in [11, 12]. We have assumed here the range of allowed values of l is so as to include all possibilities with $L_0 > 0$. Whether or not singleton contributions with $\bar{L}_0 = 0$ should be included is a subtle issue which depends on the details of the asymptotic AdS_3 boundary conditions, and is beyond the scope of this paper. The ambiguity leads to terms that depend on q but not any of the Calabi-Yau data.

As before, one can act on them with L_{-1} and generate further chiral primary states with nonzero orbital angular momenta J_ψ in AdS_3. The contribution from (4.5) to the elliptic genus is

$$\prod_{l,J_\psi \geq 0} \frac{(1-q^{l+J_\psi+1})^{n_H}}{(1-q^{l+J_\psi+1})^{2n_V+3}(1-q^{l+J_\psi+2})} = \prod_{n\geq 1}(1-q^n)M(q)^{-\chi(X)} \tag{4.6}$$

where $M(q) = \prod_{n\geq 1}(1-q^n)^n$ is the Macmahon function and $\chi(X) = 2(h^{1,1}-h^{2,1})$ is the Euler characteristic of X. We will henceforth drop the η function prefactor which does not depend on Calabi-Yau data. The net contribution from massless neutral supergravity modes including singletons is then simply

$$Z^{\mathrm{massless}}_{\mathrm{sugra}} = \prod_{n\geq 1}(1-q^n)^{-n\chi}. \tag{4.7}$$

4.3 Putting it all together

Let us now summarize and compile the results of this section into a formula for $Z_{BH} = Z_{CFT}$ The elliptic genus of the $(0,4)$ CFT as a Ramond sector trace is[6]

$$Z_{CFT}(\tau, y^A) = \mathrm{Tr}_R(-)^F q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} e^{2\pi i y^A q_A} \tag{4.8}$$

In the dilute gas expansion around $\mathrm{Im}\ \tau \to \infty$

$$\begin{aligned} Z_{CFT}(\tau, y^A) &= e^{-\pi i \tau c_L/12} Z_{\mathrm{sugra}}(\tau, y^A) \\ &= e^{-\pi i \tau c_L/12} \prod (1-e^{2\pi i \tau n})^{-n\chi} \\ &\quad \times \prod \left(1 - e^{2\pi i \tau (\frac{1}{2}q_A p^A + n + 2m_L)} e^{2\pi i q_A y^A}\right)^{(-)^{2j_R+2j_L} n(2j_R+1)N_{q_A,j_L,j_R}} \\ &\quad \times \prod \left(1 - e^{2\pi i \tau (\frac{1}{2}q_A p^A + n + 2m_L)} e^{-2\pi i q_A y^A}\right)^{(-)^{2j_R+2j_L} n(2j_R+1)N_{q_A,j_L,j_R}} \end{aligned} \tag{4.9}$$

where we take the products over the q_A charge lattice, positive integral p, n, integral or half integral j_L, j_R and $-j_{L,R} \leq m_{L,R} \leq j_{L,R}$.

[6]There are some subtleties here in the spectral flow related to the fact that the U(1) current involves membrane charges and is in a supermultiplet with the center of mass degrees of freedom [1] which we shall not try to address.

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5. Derivation of OSV

In the preceding section we computed the dilute gas approximation to the elliptic genus of the $(0,4)$ CFT, denoted Z_{BH}, as a product of terms coming from massless supergravity modes and wrapped membranes. In this section we wish to compare our result with the OSV formula [5] for the same object as the square of the topological string partition function $Z_{\rm top}$. The OSV result begins with the Bekenstein-Hawking relation and then includes all orders perurbative corrections. This is the regime in which many BPS excitations are present and is the opposite of a dilute gas. However, modular invariance relates the dilute gas to the high-temperature regime needed for comparison to OSV as follows.

Under the modular transform $\tau \to -1/\tau$, we have

$$\begin{aligned} Z_{BH} = Z_{CFT}(\tau, y^A) &= Z_{CFT}(-1/\tau, y^A/\tau) e^{-\frac{2\pi i}{\tau} y^2} \\ &= \exp\left[\frac{2\pi i}{\tau}\left(\frac{c_L}{24} - y^2\right)\right] Z_{\rm sugra}(-1/\tau, y^A/\tau) \end{aligned} \tag{5.1}$$

where $y^2 = D_{ABC} p^A y^B y^C$. In this form we can consider high temperatures $\mathrm{Im}\,\tau \to 0$ since the r.h.s. will then involve $Z_{\rm sugra}$ at low temperatures. The $(0,4)$ CFT of [1] has $c_L = 6D + c_2 \cdot P = 6D_{ABC}p^A p^B p^C + c_{2A}p^A$, The prefactor in (5.1) is then

$$\exp\left[\frac{2\pi i}{\tau}\left(\frac{c_L}{24} - y^2\right)\right] = \exp\left\{\frac{\pi^2}{\phi^0}\left[D_{ABC}p^A\left(p^B p^C - \frac{\phi^B\phi^C}{\pi^2}\right) + \frac{1}{6}c_{2A}p^A\right]\right\} \tag{5.2}$$

This is precisely $|\exp(\mathcal{F}_0^{(0)} + \mathcal{F}_1^{(0)})|^2$, where $\mathcal{F}_{0,1}^{(0)}$ denote the part of topological string amplitude that is perturbative on the world sheet, at genus 0 and 1. $Z_{\rm sugra}$ then give the rest of $|Z_{\rm top}|^2$. To see this we need to use the fundamental relation between the integral degeneracies N_{q_A, j_L, j_R} of BPS states and the coefficients of the topological string expansion found from a Schwinger computation in [13]. Indeed comparing with [13][7] we find precisely that, for purely imaginary $\tau = i\phi^0/2\pi$ and $y^A = i\phi^A/2\pi$,

$$\begin{aligned} Z_{BH} &= \mathrm{Tr}_R[(-)^F \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} e^{-\phi^A q_A - \phi^0(L_0 - \frac{c_L}{24})}] \\ &= \left| Z_{\rm top}\left(g_{\rm top} = \frac{4\pi^2}{\phi^0}, t^A = \frac{\phi^A - \pi i p^A}{\phi^0}\right)\right|^2 . \end{aligned} \tag{5.3}$$

The first line is the OSV definition of the mixed partition function and the second is the OSV relation to the square of the topological string partition function.

In conclusion we have rederived the OSV relation in all detail from an M-theory partition function on an $AdS_3 \times S^2 \times X$. In this picture the factorization into holomorphic and antiholomorphic parts has a simple origin as the contributions from M2-branes and anti-M2-branes.

[7] To see this agreement one needs to use a well-known resummation of the formulae of [13] which does not seem to be in the literature. For the readers benefit we reproduce this in the appendix.

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A. Ressumation of the GV formula

In this appendix we rearrange the expression for the topological string partition function to express it in terms of the Gopakumar-Vafa invariants rather than the degneracies N_{q_A,j_L,j_R} of the irreducible representations. Taking minus the log of the first product in (4.4) and summing over j_L, j_R, resumming and setting $g_{\rm top} = -2\pi i\tau$, $t^A = y^A + \frac{\tau}{2}p^A$ gives

$$\begin{aligned} F &= \sum_{q_A,n,m_L,j_L j_R} (-)^{2j_R+2j_L} n(2j_R+1)N_{q_A,j_L,j_R} \ln(1-e^{-g_{\rm top}(n+2m_L)}e^{2\pi i q_A t^A}) \\ &= -\sum_{q_A,n,m_L,j_L,j_R,k} (-)^{2j_R+2j_L}\frac{n}{k}(2j_R+1)N_{q_A,j_L,j_R}e^{-kg_{\rm top}(n+2m_L)}e^{2\pi i k q_A t^A} \\ &= -\sum_{q_A,j_L,j_R,k} (-)^{2j_R+2j_L}\frac{1}{k}(2j_R+1)N_{q_A,j_L,j_R}\frac{\sinh[(2j_L+1)kg_{\rm top}]}{\sinh^2\left[\frac{1}{2}kg_{\rm top}\right]\sinh[kg_{\rm top}]}e^{2\pi i k q_A t^A} \end{aligned} \tag{A.1}$$

In [4] the Gopakumar-Vafa invariants α_{r,q_A} are defined by

$$\sum_r \alpha_{r,q_A}(-)^r(2\sinh\frac{\theta}{2})^{2r} = \sum_{j_L,j_R}(-)^{2j_R+2j_L}(2j_R+1)N_{q_A,j_L,j_R}\frac{\sinh[(2j_L+1)\theta]}{\sinh[\theta]}, \tag{A.2}$$

so that

$$F = \sum_{q_A,r,k}\frac{(-)^{r-1}}{k}\alpha_{r,q_A}\left(2\sinh\frac{kg_{\rm top}}{2}\right)^{2r-2}e^{2\pi i k q_A t^A}. \tag{A.3}$$

This agrees precisely with the expression for the topological string partition function given in [4].

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